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INTERSECTION OF ASYMPTOTIC SURFACES OF THE PERTURBED EULER-POINSOT PROBLEM*

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The mutual disposition of the asymptotic surfaces (separatrices) of perturbed permanent rotations is considered in the problem of the motion of an asymmetric heavy rigid body with a fixed point in a weak gravitational field. It is shown that, for small values of the Poincaré parameter, there are always no paired separatrices, except in the Hess-Appelroth case. As the Poincaré parameter tends to zero, it is shown that an infinite number of bifurcations of the birth and disappearance of heteroclinic solutions, passing close to the three undisturbed separatrices, can be observed.

1. **The problem.** Let U be a domain in the real plane $\mathbf{R}^2\{x^1, x^2\}$, and let μ be a small parameter: $|\mu|<\varepsilon$. We consider the system

$$\frac{dx^1}{d\varphi}=\frac{\partial H}{\partial x^2},\quad \frac{dx^2}{d\varphi}=-\frac{\partial H}{\partial x^1} \tag{1.1}$$
$$(H(x^1, x^2, \varphi, \mu)=H_0(x^1, x^2)+\mu H_1(x^1, x^2, \varphi)+\ldots)$$

with Hamiltonian which is 2π-periodic with respect to time φ and analytic in the direct product

$$U\{x^1, x^2\}\times S^1\{\varphi \bmod 2\pi\}\times(-\varepsilon, \varepsilon)$$

Let the unperturbed Hamiltonian of the system

$$\frac{dx^1}{d\varphi}=\frac{\partial H_0}{\partial x^2},\quad \frac{dx^2}{d\varphi}=-\frac{\partial H_0}{\partial x^1} \tag{1.2}$$

have fixed hyperbolic points $x_1, x_2, x_3\in U$ (the points x_1 and x_3 may possibly coincide), joined by two doubly asymptotic solutions $x_1^*(\varphi), x_2^*(\varphi)$, lying entirely in the domain U:

$$\lim_{\varphi\to-\infty} x_k^*(\varphi)=x_k,\quad \lim_{\varphi\to+\infty} x_k^*(\varphi)=x_{k+1};\quad k=1, 2$$

The solutions, asymptotic as $\varphi\to-\infty$ or $\varphi\to+\infty$, to a given periodic hyperbolic solution, form two invariant surfaces, called respectively the outgoing and incoming separatrices.

System (1.2) has two pairs of coincident (twinned) asymptotic surfaces of hyperbolic periodic solutions. They are the outgoing separatrix Γ_1'' of the solution $x\equiv x_1$ and the incoming separatrix Γ_1' of the solution $x\equiv x_2$ on the one hand, and the outgoing separatrix Γ_2' of the solution $x\equiv x_2$ and the incoming separatrix Γ_2'' of the solution $x\equiv x_3$ on the other.

For small $\mu\neq 0$ the 2π-periodic hyperbolic solutions $x\equiv x_i$ $(i=1, 2, 3)$ and their asymptotic surfaces fo not vanish, but are merely slightly deformed. However, as Poincaré discovered, in the general case for small values of the parameter $\mu\neq 0$ the separatrices cease to be twinned (they split up).

Simple necessary and sufficient conditions have been obtained /1/ for the splitting, intersection, and non-intersection, of perturbed asymptotic surfaces. These results refer, however, to the mutual disposition of the perturbed separatrices in a domain which contains part of the unperturbed twinned separatrix and does not contain the unperturbed periodic solutions.

Assume that, for small $\mu>0$, the solutions $x\equiv x_i$ transform into the solutions $x=x_i(\varphi)$, and that the perturbed separatrices Γ_1', Γ_1'' and Γ_2', Γ_2'' split up and do not intersect,

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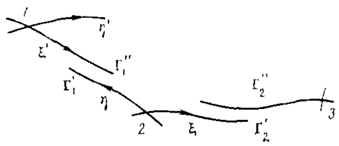

Fig.1

while Γ_k'' lie on one side of Γ_k' (the section by the plane $\varphi = \mathrm{const}$ is shown in Fig.1). We find below simple sufficient conditions for non-coincidence and intersection of the separatrices Γ_k'' for all small $\mu > 0$. Our results are used to study the asymptotic surfaces of the perturbed Euler-Poinsot problem.

2. Normal coordinates in the neighbourhood of a hyperbolic periodic solution.

We will use the "uniform version" of Mozer's theorem /1/: there exists the change of variables

$$x = \Phi(\xi, \eta, \varphi, \mu) = \Phi_0(\xi, \eta) + \mu\Phi_1(\xi, \eta, \varphi) + \ldots$$
$$\partial(x^1, x^2)/\partial(\xi, \eta) \equiv 1, \quad \Phi_0(0, 0) = x_2$$

real-analytic with respect to ξ, η, φ, μ for sufficiently small $|\xi|, |\eta|, |\mu|$, and 2π-periodic in φ, transforming system (1.1) to the normal form (the point denotes the derivative with respect to ω)

$$d\xi/d\varphi = \partial F/\partial\eta, \quad d\eta/d\varphi = -\partial F/\partial\xi \tag{2.1}$$
$$\omega = \xi\eta, \quad F(\omega, \mu) = F_0(\omega) + \mu F_1(\omega) + \ldots, \quad F_0^{\cdot}(0) = \Lambda > 0$$

It can be assumed that the outgoing separatrix $\eta = 0, \xi > 0$ coincides with Γ_2', and the incoming separatrix $\xi = 0, \eta > 0$, with Γ_1'.

Using the results of /1/, we can obtain the equation for the separatrix Γ_2'' close to Γ_2'

$$\eta = -\mu J_2(\varphi - \varphi_2)/(\Lambda\xi) + \mu^2 R_2(\xi, \varphi, \mu) \tag{2.2}$$
$$J_2(\varphi) = \int_{-\infty}^{+\infty} \{H_0, H_1\}(x_2^*(\tau - \varphi), \tau)\, d\tau$$
$$x_2^*(\tau + \varphi_2) = \Phi_0(\xi \exp(\Lambda\tau), 0) \tag{2.3}$$

Here and below, $R_1, \ldots, R_7$ are analytic functions, and $C_1, \ldots, C_8$ are constants; condition (2.3) serves to define φ_2.

The similar equation for Γ_1'' close to Γ_1' is

$$\xi = \mu J_1(\varphi - \varphi_1)/(\Lambda\eta) + \mu^2 R_1(\eta, \varphi, \mu) \tag{2.4}$$
$$J_1(\varphi) = \int_{-\infty}^{+\infty} \{H_0, H_1\}(x_1^*(\tau - \varphi), \tau)\, d\tau$$
$$x_1^*(\tau + \varphi_1) = \Phi_0(0, \eta \exp(-\Lambda\tau)) \tag{2.5}$$

Let ξ', η', φ be the coordinates in the neighbourhood of the perturbed periodic solution $x = x_1(\varphi)$, similar to coordinates ξ, η, φ, and let Λ' be a quantity similar to Λ. To be specific, let ξ', η' be chosen in the way shown in Fig.1. (The case shown in Fig.1 can only hold if $J_1(\varphi) \geqslant 0, J_2(\varphi) \leqslant 0$.)

By composing the Birkhoff transformation with powers of the mapping over the period we can continue the coordinates ξ, η and ξ', η' in a neighbourhood V_1 of the separatrices Γ_1', Γ_1'', which does not contain the perturbed solutions $x = x_k(\varphi)$. In the domain V_1 it is convenient to transform from coordinates ξ, η and ξ', η' to coordinates ω, φ_1 and ω', φ_2', where $\omega = \xi\eta$, $\omega' = \xi'\eta'$, and φ_2' is expressible in terms of ξ' by an expression exactly similar to (2.3). Using (2.2)-(2.5), we find expressions for transforming from one coordinate system to the other in the domain V_1:

$$\Lambda\omega = \Lambda'\omega' + \mu J_1(\varphi - \varphi_2') + R_3(\omega', \varphi, \varphi_2', \mu) \tag{2.6}$$
$$\varphi_1 = \varphi_2' + R_4(\omega', \varphi, \varphi_2', \mu)$$

where the series expansion of R_3 in ω', μ starts with not lower than quadratic terms, and the expansion of R_4 with not lower than linear terms.

Notice that: 1) $\omega, \omega', J_1(\varphi - \varphi_2')$ is independent of the choice of coordinates ξ, η and ξ', η', and depends only on the point of the domain V_1 and the parameter μ; 2) Eqs.(2.2), (2.4) are obtained from (2.6) if we put $\omega = 0$ or $\omega' = 0$.

We can express φ_1, φ_2 in terms of ξ, η by

$$\varphi_1 = C_1 - \Lambda^{-1} \ln \eta, \quad \varphi_2 = \Lambda^{-1} \ln \xi + C_2 ,$$

which follow from Eqs. (2.3) and (2.5).

In the neighbourhood V_1 of the separatrices Γ_1', Γ_1'' we shall use coordinates $\omega', \varphi_2', \varphi$, and in the similar neighbourhood V_2 of the separatrices Γ_2', Γ_2'', the coordinates, $\omega, \varphi_2, \varphi$. Let $(\omega, \varphi_2', \varphi)$ be the coordinates of a point close to separatrices Γ_1', Γ_1'' (inside the domain V_1), where $\omega' = \mu J'$, where $J' = O(1)$. Then in $(\omega, \varphi_1, \varphi)$ coordinates this point becomes

$$\omega = \mu J, \ \varphi_1 = \varphi_2' + O(\mu) \qquad (2.7)$$
$$\Lambda J = \Lambda' J' + J_1(\varphi - \varphi_2') + O(\mu)$$

If $J \geqslant C_3 > 0$ (all the expressions are similar for the case $J \leqslant C_3 < 0$), then, after a time $\Delta\varphi = 2\pi n$ (n iterations of the Poincaré mapping), where $n = [-(2\pi\Lambda)^{-1} \ln \mu]$, the point $(\xi, \eta, \varphi) \sim (\omega, \varphi_2', \varphi)$ becomes the point $(\xi_1, \eta_1, \varphi + \Delta\varphi) \sim (\omega, \varphi_2, \varphi)$, close to Γ_2' (inside the domain V_2), where

$$\xi_1 = \xi \exp(2\pi w) = O(1), \ \eta_1 = O(\mu) \qquad (2.8)$$
$$w = F^{\cdot} = \Lambda + \mu\alpha + \mu^2 R_5(J, \mu)$$
$$\alpha = F_0^{\cdot\cdot}(0) J + F_1^{\cdot}(0)$$
$$\varphi_2 = \Lambda^{-1} \ln \xi_1 + C_2 = \Lambda^{-1} \ln \omega - \Lambda^{-1} \ln \eta + C_2 + 2\pi n + 2\pi n \Lambda^{-1}\mu\alpha +$$
$$2\pi n \Lambda^{-1}\mu^2 R_5(J, \mu) = \varphi_1 + C_4 + \Lambda^{-1} \ln \mu + 2\pi n + 2\pi n \Lambda^{-1}\mu\alpha + \Lambda^{-1} \ln J + 2\pi n \Lambda^{-1}\mu^2 R_5(J, \mu)$$

3. Non-coincidence of the separatrices Γ_1'', Γ_2''.

Theorem 1. The separatrices Γ_1'', Γ_2'' do not coincide for any sufficiently small $\mu > 0$ if at least one of the following conditions holds:

1°. $\frac{d}{d\varphi} \ln J_1(\varphi) \geqslant \Lambda$ or $\frac{d}{d\varphi} \ln(-J_2(\varphi)) \leqslant -\Lambda$

for any φ (this is the case, in particular, if $J_1(\varphi) = 0$ or $J_2(\varphi) = 0$ for some φ).

2°. The domains of variation of the functions J_1 and $-J_2$ are not the same.

3°. One of functions J_1, J_2 has no branching points in the complex plane, while the other is not equal to a constant and has a zero or pole on its Riemann surface. (These conditions are satisfied, e.g., by real trigonometric polynomials.)

4°. $F_0^{\cdot\cdot}(0) \neq 0$ and at least one of functions J_i is not constant.

Some other criteria can also be obtained for non-coincidence and intersection of the separatrices Γ_1'', Γ_2''.

Proof. Let us find the parametric equations for the separatrix Γ_1'' close to Γ_2', Γ_2''. From (2.4) and (2.8) we have

$$\psi_2 = \psi_1 - C_4 - \Lambda^{-1} \ln \mu - 2\pi n - \Lambda^{-1} \ln \Lambda^{-1} - \qquad (3.1)$$
$$\Lambda^{-1} \ln J_1(\psi_1) + O(\mu \ln \mu)$$
$$\omega = \Lambda^{-1}\mu J_1(\psi_1) + O(\mu^2), \ \psi_k = \varphi - \varphi_k$$

The term $O(\mu \ln \mu)$ on the right-hand side of (3.1) has the form

$$-2\pi n \Lambda^{-1}\mu\alpha(\varphi - \varphi_1) + \mu R_6(\varphi, \varphi_1, \mu) +$$
$$n\mu^2 R_7(\varphi, \varphi_1, \mu)$$
$$\alpha(\psi) = F_1^{\cdot}(0) + \Lambda^{-1} F_0^{\cdot\cdot}(0) J_1(\psi)$$

If, for some $\mu > 0$, the separatrices Γ_1'', Γ_2'' coincide, then φ_2 with fixed φ is a single-valued smooth function of φ_1; similarly, φ_1 is a smooth function of φ_2. The condition $d\varphi_2/d\varphi_1 \geqslant \delta > 0$ must therefore be satisfied. Calculation gives

$$d\varphi_2/d\varphi_1 = 1 - \Lambda^{-1} d \ln J_1(\varphi - \varphi_1)/d\varphi - 2\pi n \Lambda^{-1}\mu d\alpha(\varphi - \varphi_1)/d\varphi + O(\mu)$$

Hence criterion 1° follows.

The curve cut out by the plane $\varphi = \text{const}$ on the separatrix Γ_i'' is the image under a power of the Poincaré mapping of the piece of it lying close to separtrix x_i^*. From (2.2), (2.4) we have

$$| J_1(\varphi - \varphi_1) + J_2(\varphi - \varphi_2) | \leqslant C_5 |\mu| \qquad (3.2)$$

Hence criterion 2° follows at once.

Now let $F_0^{\cdot\cdot}(0) \neq 0$, $dJ_1/d\psi \neq 0$ at the point $\psi = \psi_0$. Then, $(d\alpha/d\psi)_{\psi=\psi_0} \neq 0$, and it can be shown that inequality (3.2) must be violated for small $\mu > 0$. Hence criterion 4° follows.

Let p and q be the canonical coordinates in the neighbourhood of x_2 at which the Hamiltonian

$$H_0 = \lambda pq + \sum_{\alpha+\beta\geq 3} H_{\alpha\beta} p^\alpha q^\beta .$$

On performing one step of the Birkhoff transformation, we can obtain the expression

$$^1/_2 F_0^{\cdot\cdot}(0) = -3\lambda^{-1}(H_{03}H_{30} + H_{12}H_{21}) + H_{22} .$$

If there exist arbitrarily small positive values of μ at which the separatrices Γ_1'', Γ_2'' are twinned, then, on passing to the limit as $\mu \to 0$ in the appropriate sequence convergent to zero, we obtain $-J_2(\psi_2) = J_1(\psi_1)$, where

$$\psi_2 = \psi_1 + C_6 - \Lambda^{-1}\ln J_1(\psi_1).$$

We can similarly express ψ_1 in terms of ψ_2:

$$\psi_1 = \psi_2 + C_7 + \Lambda^{-1}\ln(-J_2(\psi_2)).$$

Let $f(\psi) = J_1(\psi)$, $g(\psi) = -J_2(\psi)$, and let f have a zero w of order $m > 0$ on its Riemann surface, which lies above $z_0 \in \mathbf{C}$ (in other cases, all the arguments are similar). Then w has a neighbourhood on the Riemann surface which projects one-to-one into the neighbourhood U of point z_0 in the complex plane. Hence we need only consider in the set U one branch of the function f, which is connected with the function g by the equation

$$f(z) = g(z + X),\ X = C_6 - \Lambda^{-1}\ln f(z)$$

If z is close to z_0, then $\mathrm{Re}\,X$ is close to $+\infty$.

There is a C_8 such that, for u that satisfies the condition $\mathrm{Re}\,u > C_8$, there is a solution $z \in U$ of the equation $z + X = u$.

For, we can rewrite the last equation as

$$f(z)\exp(-\Lambda z) = \exp(-\Lambda(u - C_6)) \tag{3.3}$$

By the theorem on local inversion of analytic functions, there exists $\rho > 0$ such that, if $|\exp(-\Lambda(u - C_6))| < \rho$, then Eq.(3.3) has a solution $z \in U$ which tends to z_0 as $\mathrm{Re}\,u \to +\infty$. Thus, as $\mathrm{Re}\,u \to +\infty$ $g(u) = f(z) \to f(z_0) = 0$. Since $g(u)$ is 2π-periodic, $g \equiv 0$. We have a contradiction. Criterion 3 is proved.

4. Application to the perturbed Euler-Poinsot case. We consider the motion of an asymmetric rigid heavy body about a fixed point. Let $a < b < c$ be the reciprocals of the principal moments of inertia of the body; the Poincaré parameter μ is the product of the weight of the body and the distance of the centre of gravity to the point of suspension, X_0, Y_0, Z_0 are the direction cosines of the radius vector of the centre of gravity in the principal axes of inertia, connected with the fixed point, and H is the constant area.

If the total energy level $h > 0$ is fixed, we can pass by means of isoenergetic reduction to the reduced system (1.1) /1/, where $x^1 = l$, $x^2 = L$, $\varphi = g$ are the Andoyer-Deprit canonical variables. With $\mu = 0$ system (1.1) has the fixed points

$$\gamma_1 : (L = 0,\ l = \pi \bmod 2\pi),\ \gamma_2 : (L = 0,\ l = 0 \bmod 2\pi)$$

connected by the doubly asymptotic solutions. Study of the splitting of the separatrices with $\mu \neq 0$ was started in /2/ (in the special case $X_0 = Z_0 = 0, Y_0 \neq 0$), and was completed in /1/.

It was found that, with certain values of the problem parameters, the separatrices split up and do not intersect. Nevertheless, Ziglin showed, by applying to a sequence mapping given in a domain of the ring $(L;\ l \bmod 2\pi)$, Mozer's theorem on invariant curves, and using simple arguments, connected with the presence of an invariant area, that the following can easily be proved: for all values of the problem parameters, except for the Hess-Appelroth case, for sufficiently small $\mu \neq 0$, there are at least two double asymptotic (homoclinic) solutions for each disturbed periodic solution γ_i. In the Hess-Appelroth case, there are no such solutions.

It has remained unclear whether some of these homoclinic solutions can lie on twinned (for certain small $\mu \neq 0$) asymptotic surfaces. This can only occur in the situation studied in Sect.3. In this problem, the improper integrals $J_i(\varphi)$, taken along unperturbed double asymptotic solutions, are trigonometric non-constant polynomials /1/. Hence, by criterion 3°, the asymptotic surfaces do not coincide for any small $\mu \neq 0$.

Take three doubly asymptotic solutions $x_i^*(\varphi)$, $i = 1, 2, 3$ (Fig.2), chosen so that the points $x_i^*(0)$ are equidistant from the fixed points γ_i. Let $J_i(\varphi)$ be the corresponding improper integrals. Using the results of /1/, we find after calculations that

$$J_j(\varphi) = (-1)^j a_0 Y_0 + a_y Y_0 \cos\varphi - ((-1)^j a_x X_0 + a_z Z_0)\sin\varphi,$$
$$j = 2, 3;$$
$$a_0 = H/h,\ a_x = k\sqrt{c - b}/\mathrm{sh}\,{}^1/_2\pi\beta$$
$$a_y = k\sqrt{c - a}/\mathrm{ch}\,{}^1/_2\pi\beta,\quad a_z = k\sqrt{b - a}/\mathrm{sh}\,{}^1/_2\pi\beta$$
$$k = \pi G_0^{-1}[1 - (H/G_0)^2]^{1/2}[(b - a)(c - b)(c - a)]^{-1/2}$$
$$h = {}^1/_2 bG_0^2,\ \Lambda = \Lambda' = b^{-1}[(b - a)(c - b)]^{1/2} = \beta^{-1}$$

(h is a fixed energy constant). The integral $J_1(\varphi)$ is obtained from $J_2(\varphi)$ by the replacement $X_0 \to -X_0, Y_0 \to -Y_0$. From the inequality $a^{-1} < b^{-1} + c^{-1}$ for the moments of inertia, we have $\Lambda < 1$.

We denote the problem parameters by

$$pr = (a, b, c, X_0, Y_0, Z_0, H/G_0)$$

Theorem 2. There exist domains S_i $(i = 1, 2, 3)$ in the parameter space such that: 1) with $pr \in S_1 \cup S_2 \cup S_3$ and all small $\mu > 0$, the perturbed separatrices split up, do not intersect, and are located as shown in Fig.3;

2) for $pr \in S_1$ and all small $\mu > 0$, the outgoing separatrix Γ_1 and the incoming separatrix Γ_2 do not intersect close to the unperturbed separatrices x_i^*;

3) for $pr \in S_2$ and all small $\mu > 0, \Gamma_1$ and Γ_2 intersect close to the unperturbed separatrices x_i^*;

4) for $pr \in S_3$ there are sequences of positive numbers $\mu_n^+ \to 0, \mu_n^- \to 0, n \to \infty$, such that, with $\mu = \mu_n^-$ Γ_1 and Γ_2 intersect close to x_i^*, and with $\mu = \mu_n^+$ they do not intersect.

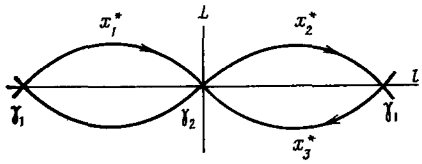

Fig.2

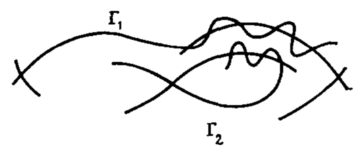

Fig.3

In short, for $pr \in S_3$, as the positive μ tends to zero, we observe an infinite number of bifurcations of the birth and disappearance of heteroclinic solutions, passing close to x_i^*. This is also true for $pr \in S_2$, though in this case these heteroclinic solutions do not all vanish for every small $\mu > 0$.

Proof. The picture of the separatrices in Fig.3 holds if $J_1(\varphi) > 0, J_2(\varphi) < 0, J_3(\varphi) > 0$ for all φ. In the neighbourhood of the perturbed periodic solution γ_i we choose normal coordinates ξ_i, η_i, and take $\omega_i = \xi_i\eta_i$. In the neighbourhood of x_i^* we link with $|\xi_{i \bmod 2}|$ the phase φ_i in accordance with relations (2.3). Let $\psi_i = \varphi - \varphi_i$.

We use expressions (2.7), (2.8), (3.1). Close to x_2^* the separatrix Γ_1 is given by the parametric equations

$$\psi_2 = \psi_1 - t - \Lambda^{-1} \ln J_1(\psi_1) + O(\mu \ln \mu) \tag{4.1}$$
$$(t = C_4 + \Lambda^{-1} \ln \mu + 2\pi n + \Lambda^{-1} \ln \Lambda^{-1})$$

where $\omega_2 = \Lambda^{-1}\mu J_1(\psi_1) + O(\mu^2)$, or

$$\omega_3 = \Lambda^{-1}\mu (J_1(\psi_1) + J_2(\psi_2)) + O(\mu^2) \tag{4.2}$$

The parts of Γ_1, where $-(J_1(\psi_1) + J_2(\psi_2)) \geqslant \delta > 0$, are located close to x_2^*, and in time $2\pi n$ transform into parts located close to x_3^* and given by Eqs.(4.2) and

$$\psi_3 = \psi_2 - t - \Lambda^{-1} \ln(-J_1(\psi_1) - J_2(\psi_2)) + O(\mu \ln \mu) \tag{4.3}$$

(by the symmetry of the unperturbed problem, the constant C_4 is the same in both equations). Close to x_2^* the separatrix Γ_2 is given by the equation

$$\omega_3 = -\Lambda^{-1}\mu J_3(\psi_3) + O(\mu^2)$$

By choosing suitable a, b, c, H, G_0, we can give any pre-assigned values to the quantities $a_0, a_y \neq 0, 0 < \Lambda < 1$. Let $X_0 = Z_0 = 0, Y_0 = 1$.

If $J \equiv J_1(\psi_1) + J_2(\psi_2) + J_3(\psi_3) > 0$ for any ψ_i, then the separatrices Γ_1, Γ_2 do not intersect close to x_3^* for small $\mu > 0$, or therefore, close to x_1^*, x_2^*. For this, it suffices to require that $3|a_y| < |a_0|$.

The piece of Γ_2 close to x_2^* is given by the equations

$$\omega_2 = -\Lambda^{-1}\mu (J_3(\psi_3) + J_2(\psi_2)) + O(\mu^2) \tag{4.4}$$
$$\psi_2 = \psi_3 + t + \Lambda^{-1} \ln J_3(\psi_3) + O(\mu \ln \mu)$$

We fix a_0, a_y, ψ_2. In any interval in $S^1 = \mathbf{R}/(2\pi\mathbf{Z})$ for sufficiently large Λ^{-1}, there exists a solution ψ_1 of Eq.(4.1) mod 2π and a solution ψ_3 of Eq.(4.4) mod 2π (since functions J_1, J_3 are not locally constant). It can be seen that the following holds: for large Λ^{-1}, in the neighbourhood of separatrix x_2^* which does not contain solutions γ_i, there exist branches of separatrices Γ_1 and Γ_2, $o(\mu)$-close to the surfaces given by the equations

$$\omega_2 = \Lambda^{-1}\mu J_1(\psi_1), \quad \omega_2 = -\Lambda^{-1}\mu(J_2(\varphi-\varphi_2)+J_3(\psi_3))$$

where ψ_1, ψ_3 are previously chosen numbers. Hence, if there are numbers ψ_1, ψ_3, such that J changes sign as ψ_2 varies, then, for sufficiently large Λ^{-1} and small $\mu > 0$ $(0 < \mu < \mu(pr))$, the separatrices Γ_1, Γ_2 have an infinitely large number of distinct lines of intersection, i.e., heteroclinic solutions. Hence there exists $pr \in S_2$.

We now fix Λ and specify a positive $\varepsilon < 1/2$, while we take $\delta = |a_y/a_0|$ sufficiently small $(\delta < \delta(\varepsilon))$. We choose X_0, Y_0, Z_0 in such a way that

$$a_xX_0 + a_zZ_0 = 0$$
$$|a_0Y_0|^{-1}(a_zZ_0 - a_xX_0) = 1/2 + \varepsilon.$$

It can be assumed without loss of generality that $a_0Y_0 = -1$.

Let $\alpha = (1-2\varepsilon)/(1+2\varepsilon)$. The quantity $J + O(\mu)$ can be equal to zero, provided that $\sin\psi_1$ and $\sin\psi_3$ exceed $\alpha + O(\delta) + O(\mu)$, i.e., that ψ_1, ψ_3 lie in the intervals $(2\pi m + \Phi, 2\pi m + \pi - \Phi)$, where $\Phi = \arcsin\alpha + O(\delta) + O(\mu)$. In this case

$$-(J_1(\psi_1) + J_2(\psi_2)) > 1/2 - \varepsilon + O(\delta) + O(\mu).$$

By (4.1) and (4.3),

$$\psi_3 - \psi_1 = -2t - 2\Lambda^{-1}\ln 1/2 + O(\varepsilon) + O(\delta) + O(\mu\ln\mu).$$

On the other hand, the difference $\psi_3 - \psi_1$ must lie in one of the intervals $(2\pi m + 2\Phi - \pi, 2\pi m + \pi - 2\Phi)$. If ε is small $(\varepsilon < \varepsilon(\Lambda), \delta < \delta(\varepsilon), \mu < \mu(pr))$, this condition must be violated in a sequence $t_n \to -\infty, n \to \infty$, to which corresponds the sequence μ_n^+.

The sequence μ_n^- will correspond to the t which satisfy the condition

$$2t + \Lambda^{-1}\ln(1/4 - \varepsilon^2) = 0 \bmod 2\pi.$$

For, in this case, with $\sin\psi_1 = 1$ we have

$$\psi_3 = \psi_1 + O(\delta) + O(\mu\ln\mu) \bmod 2\pi$$
$$J + O(\mu) = -2\varepsilon + O(\delta) + O(\mu)$$

This last expression is negative if $\delta < \delta(\varepsilon)$ and $\mu < \mu(pr)$. Then, the separatrices Γ_1 and Γ_2 intersect close to x_3^*, and hence, close to x_1^*, x_2^*. Thus there exists $pr \in S_3$.

Notice in conclusion that all the points in the problem parameter space, close to those chosen in the proof, likewise have the necessary property. Hence we can choose entire domains as the sets S_i.

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